axis of revolution. If arbitrary loading is represented by Fourier series, additional constraint conditions for the degrees of freedom of the axis nodes are derived from the consideration that, for any axis point, the strains associated with the assumed linear displacement field should not become singular.

It is the purpose of this Comment to point out that these constraint conditions follow directly from the general kinematical compatibility requirements, i.e., from the condition that the displacement field be continuous. Corresponding constraint conditions may be applied[2,3] to the analysis of axisymmetric solids using not only linear displacement but also quadratic and cubic displacement triangular ring elements.

Furthermore, if instead of the third of Belytschko's Eq. (3)[1] we use

$$u_\theta(r, z, \theta) = \sum_n u_\theta{}^n(r, z)(-\sin n\theta) + \sum_n u_\theta{}^{-n}(r, z)\cos n\theta \qquad (1)$$

the element stiffnesses and associated matrices will be the same for the symmetric and antisymmetric components of each harmonic, i.e., for n and $-n$ (with the exception of the zeroth harmonic). Thus the effort to set up the stiffness matrices is considerably reduced.

References

[1] Belytschko, T., "Finite Elements for Axisymmetric Solids under Arbitrary Loadings with Nodes on Origin," *AIAA Journal*, Vol. 10, No. 11, Nov. 1972, pp. 1532–1533.

[2] Buck, K. E., "Zur Berechnung der Verschiebungen und Spannungen in Rotationskoerpern unter beliebiger Belastung," Dr.-Ing. thesis, 1970, Univ. of Stuttgart, West Germany.

[3] Argyris, J. H., Buck, K. E., Grieger, I., and Mareczek, G., "Application of the Matrix Displacement Method to the Analysis of Pressure Vessels," *Transactions of the ASME*, Ser. B; *Journal of Engineering for Industry*, Vol. 92, No. 2, May 1970, pp. 317–329.

Reply by Author to K. E. Buck

TED BELYTSCHKO*

University of Illinois at Chicago, Chicago, Ill.

REGARDLESS of whether the constraint conditions, Eqs. (5) are derived from continuity or the boundedness of the strains, the resulting conditions must be directly imposed on the displacement fields of elements with nodes on the origin to avoid singularities in the stiffness matrices. Reference 2 of the Comment does not mention this.

It should be added that maintaining the boundedness of all terms of the stiffness matrix is not essential in static analysis. If the singular terms are not omitted, the numerical integration of these terms then yields elements in the stiffness equations which are an order of magnitude larger than the others; these larger terms are essentially linear combinations of the constraints. However, this is a rather unsound manner for treating these terms and fails totally in the explicit integration of the transient equations. In that case, the stability limit on the time step is inversely proportional to the largest element in the stiffness matrix. Thus the inclusion of the unbounded stiffness matrix terms leads to prohibitively small stability limits and the explicit integration of the transient equations is impossible. Hence, the enforcement of the constraints on the element displacement field as given in Ref. 1 is mandatory for any application of the stiffness matrix to explicit transient solutions and also preferable in static solutions.

Reference

[1] Belytschko, T., "Finite Elements for Axisymmetric Solids under Arbitrary Loadings with Nodes on Origin," *AIAA Journal*, Vol. 10, No. 11, Nov. 1972, pp. 1532–1533.

Received April 20, 1973.

Index categories: Structural Static Analysis; Structural Dynamic Analysis.

* Assistant Professor of Structural Mechanics.

Closed-Form Lift and Moment for Osborne's Unsteady Thin-Airfoil Theory

NELSON H. KEMP*

Avco Everett Research Laboratory Inc., Everett, Mass.

IN Ref. 1 Osborne presented an approximate theory for the unsteady motion of a two-dimensional thin airfoil in subsonic flow. As applications, he wrote the lift and moment for an airfoil subject to three oscillating upwash distributions whose chordwise dependence can be expressed in a cosine series. In only one of these three cases was the lift and moment written in closed form. In the other two, they involved infinite series of products of Bessel functions.

One purpose of this Comment is to point out that all these series can be summed, so that closed form expressions for the lift and moment can be obtained in all the cases considered by Osborne, thereby simplifying considerably their use in numerical calculation. For example, these simplifications would have been of value in Osborne's recent discussion of compressibility effects in unsteady interactions between the blade rows of turbomachines.[2]

A second purpose is to present, also in closed form, the lift and moment for the case of pitching oscillations, to complement the plunging oscillation case given by Osborne.

The notation used will be that of Ref. 1, where the most general type of upwash considered has the form

$$v(x, t) = v_0 \exp(i\nu t - i\mu x/V) \qquad (1)$$

This is referred to by Osborne as "Kemp-type upwash" because it was first introduced in Ref. 3. In terms of the parameters

$$\lambda = \mu c/V, \quad \omega = \nu c/V, \quad \omega^i = \omega/\beta^2, \quad \lambda^i = M^2\omega^i, \quad \Lambda = \lambda + \lambda^i \qquad (2)$$

and the Bessel function abbreviations

$$J = J_0 - iJ_1, \quad C = K_1/(K_0 + K_1), \quad S(z) = [iz(K_0 + K_1)]^{-1} \qquad (3)$$

the lift and nose-up moment for upwash (1) is given by Osborne[1] in Eqs. (29) and (30) as

$$L(t) = 2\pi\beta^{-1}\rho_\infty cVv_0 e^{i\nu t}\Big\{J(\Lambda)[C(\omega^i)J(\lambda^i) + iJ_1(\lambda^i)] + i(\omega^i/\Lambda)[J_0(\lambda^i)J_1(\Lambda) - J_1(\lambda^i)J_0(\Lambda)] - 2i[(\Lambda - \omega^i)/\Lambda^2]\sum_{n=1}^{\infty} nJ_n(\lambda^i)J_n(\Lambda)\Big\} \qquad (4)$$

Received March 28, 1973.

Index categories: Subsonic and Transonic Flow; Nonsteady Aerodynamics.

* Principal Research Scientist. Associate Fellow AIAA.